Introduction to Mathematical Induction

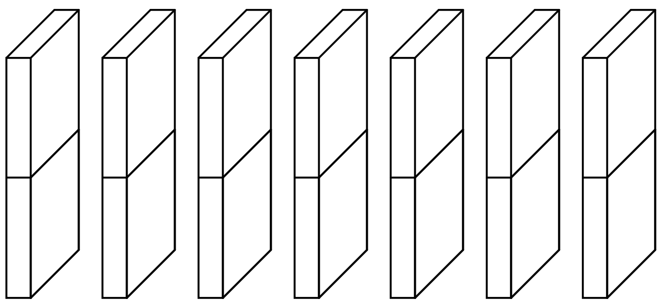
An Independent Study
Student Guide

Introduction

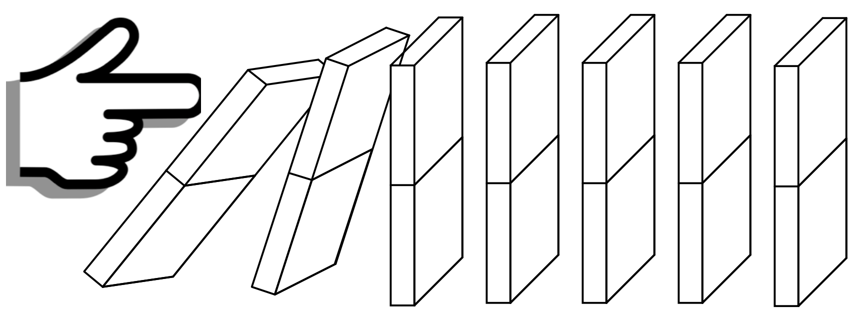
Mathematical induction is a very powerful tool for mathematical proof writing that can be found in almost all branches of mathematics. Unfortunately, when it is first presented, mathematical induction can be quite confusing and many students try to "memorize" the steps instead of understanding the process. The purpose of this booklet is to highlight the importance of the parts of the induction process by using some concrete examples and analogies.

Falling Dominoes

Look at the set of dominoes below.



From experience, we can tell that if the dominoes are properly placed and the first domino is pushed, a cascade effect will occur.



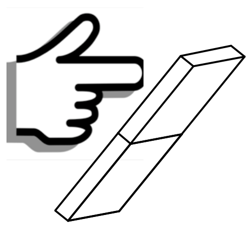
Now imagine that we have a whole gymnasium full of aligned dominoes. In your mind, you can probably picture thousands and thousands of dominoes falling over in a perfect series. This, of course, is based on your prior knowledge. Perhaps you viewed a similar demonstration on television.

How can we prove that all those dominoes will fall?

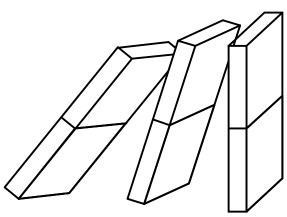
If we sit back and think about it though, in order for all the dominoes to fall, a few things must take place:

- If one domino falls it must be able to knock over the next domino in the line.
- Someone needs to push the first domino down.

Step A
We know that if we push one domino over, it will fall



Step B
We know that if each falling domino is placed in the proper position, it will knock over the next domino.



Step C
Based on intuition, if both Step A and Step B occur, all the dominoes will fall.

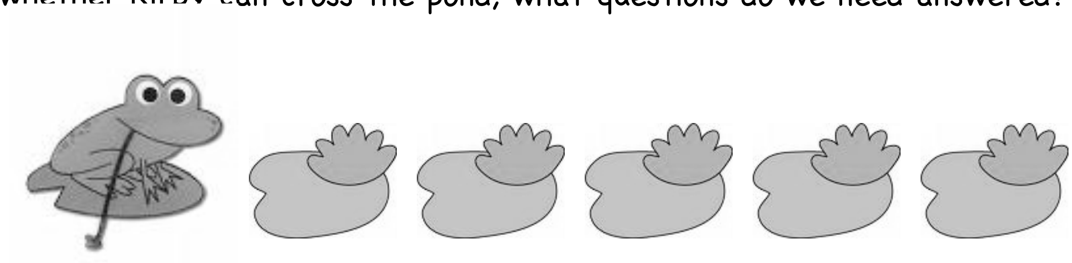
BUT, this will happen if an only if Step A and Step B occur. One step is just as important as the other. Imagine a gymnasium full of dominoes and nobody pushes the first one down (Step A). Now imagine if all the dominoes are not lined up properly (Step B).

The final result, a gymnasium full of fallen dominoes, will only occur if both Step a and Step B occur.

As we will see, the domino effect and mathematical induction have a number of similarities but before we get to mathematical, let's look at another simple example.

Kirby and the Lily Pad

Suppose Kirby the frog wants to cross a pond. In front of him, there is an infinite set of identical lily pads all equally spaced. In order to determine whether Kirby can cross the pond, what questions do we need answered?



- Can Kirby reach the first pad and will it hold his weight?
- Can he jump the space between lily pads?

Suppose we know two pieces of information:

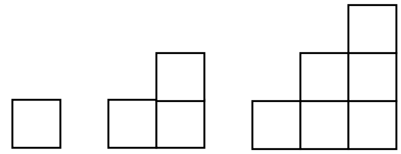
- Kirby can reach the first lily pad and it will hold his weight
- When Kirby lands on the n^{th} lily pad he is able to jump to the $n^{th}+1$ pad

From these two pieces of information combined with the fact that the identical lily pads are equally spaced, we can conclude that Kirby will be able to cross the pond.

Notice that in order to make the final conclusion, both pieces of information are required. One statement is just as important as the other.

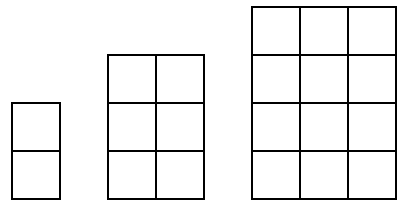
Investigation 1 – Consecutive Natural Numbers

1. On the graph paper provided, copy these figures and draw the next three diagrams in the sequence. Write their sums below each diagram.



1   1+2   1+2+3

2. If we duplicate each drawing from step 1, rectangles can be created. Do the same for the diagrams that you created on the graph paper. Write the appropriate sum underneath.



2(1)   2(1+2)   2(1+2+3)

3. Complete the following chart. If you observe the pattern, no further drawings are necessary.

Value of *n*	Value of 2(1+2+…*n*)	Dimensions of Rectangle Formed by Two Blocks	Are Columns 2 and 3 Equal?
1	2(1)	1x2	yes
2	2(1+2)	2x3	yes
3	2(1+2+3)	3x4	yes
4			
5			
6			
7			
. . . 10			
. . . 12			
. . . k			Cannot check but observed a pattern.

4. Taking note of the 2nd and 3rd column in the chart, can you make a conjecture?

Making a conjecture based on patterns can be a logical first step in mathematics. Once a conjecture is made it would be nice to prove that our conjecture holds true in all situations, or in our case, for all natural numbers ***n***.

The proof we will use is called mathematical induction. The process works much the same as the story of the falling dominoes or Kirby the frog. Recall that in both stories, two important statements where needed in order to make a reasonable conclusion.

In mathematical induction the first statement (domino will fall/first lily pad) is called the ***basis step.***

The second statement (falling domino hits another domino/Kirby can jump from one pad to the next) is called the inductive step.

In our situation, we need to show that the conjecture works for n=1. This will be the basis step.

5. Complete the formula for n=1 and determine if the statement is true.

_________________=_________________

Next we need to determine that if the statement works for n=k will it work for n=k+1. This is similar to the statement "if the n^{th} domino falls, so will the $n^{th}+1$".

First we need to assume that the k^{th} domino actually exists. In our conjecture the k^{th} situation can be found by letting n=k

$$2(1+2+3+\ldots k) = k(k+1)$$

We need to assume that this is true or we can't move on. Similarly, we needed to assume that there was a k^{th} lily pad or we couldn't move on.

Now that we know the k^{th} lily pad exists we need to ensure that we can make it to the next lily pad, k+1.

Similarly, if we assume that our conjecture works for n=k, we need to show that it is still true for n=k+1.

$$2(1+2+3+\ldots k+(k+1)) = (k+1)(k+2)$$

From our above assumption, we know that

$$2(1+2+3+\ldots k) = k(k+1)$$

or

$$(1+2+3+\ldots k) = \frac{k(k+1)}{2}$$

Revisiting our statement,

$$2(\underbrace{1+2+3+\ldots k}_{\frac{k(k+1)}{2}}+(k+1)) = (k+1)(k+2)$$

$$2\left(\frac{k(k+1)}{2}+(k+1)\right) = (k+1)(k+2)$$

Expanding the left side,

$$k^2+k+2k+2 = (k+1)(k+2)$$

$$k^2+3k+2=(k+1)(k+2)$$

Factoring the left side,

$$(k+1)(k+2) = (k+1)(k+2)$$

The left side equals the right side thus it must be a true statement.

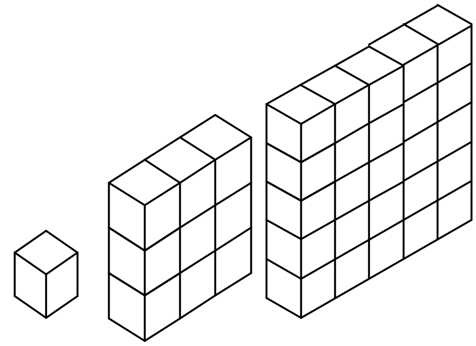
Recap:

- We showed that n=1 was true for our conjecture
- We assumed that n=k was true
- Using our n=k assumption we proved that n=k+1 was true for our conjecture.

Can you see the similarities between this mathematical proof and the domino effect?

Investigation 2 – Squares of Odd Natural Numbers

In this activity we will examine the sum of squares of consecutive odd natural numbers. The squares of odd numbers will be represented by cubes of width 1.



1^2 3^2 5^2

1. Take six unit cubes and form a 1x2x3 rectangular solid. Draw a picture of the solid with 3 as the height. From here we can see that $6(1^2) = 1\text{x}2\text{x}3$. Begin to fill in the chart below.

2. Next take your 1x2x3 solid that you created in step 1 and 6 representation for 3^2 (see pic on the right), and build a new rectangular solid (see fig. A). Write the dimension of the new solid by in ascending order. Thus $6(1^2)+6(3^2) = 6(1^2+3^2) =$ ________________
Continue filling out the chart.

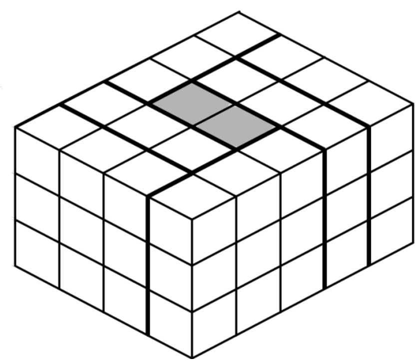


Fig A

3. The next solid is built by adding six representations of 5^2 to the figure that was created in step 2. For this figure, $6(1^2)+6(3^2)+6(5^2)$, the dimensions in ascending order are ____________________.
Continue filling out the chart.

4. Complete the chart without drawing the figures for n=7,9, or *k*. Make a conjecture.

Value of n	Sum $6(1^2+3^2+\ldots n^2)$	Dimensions of Solid	Are Columns 2 and 3 Equal?
1			
3			
5			
7			
9			
. . . k			

5. Consider the formula $6(1^2+3^2+5^2+\ldots+(n-2)^2+n^2)=n(n+1)(n+2)$, where n is any positive odd integer. Complete the following.

The formula for n=1 is ___________________=__________________.

The formula for n=k is ___________________=__________________.

The formula for n=k+2 is ___________________=__________________.

Now we will prove that the formula is true for all positive odd integers n using the principles of mathematical induction.

Step 1 (Basis): Show that the formula

$$6(1^2+3^2+5^2+\ldots+(n-2)^2+n^2)=n(n+1)(n+2),$$

is true for n=1.

Step 2 (Assumption): Assume that the formula is true for n=k. (Write out the formula)

Step 3 (Induction Step): Using the assumption from step 2, show that the formula holds true for n = k+2 (See Investigation 1 for help)

Investigation 3 – Squares of Even Natural Numbers

Using a mathematical induction proof similar to Investigation 2, show that for any positive even integer n,

$$6(2^2+4^2+6^2+\ldots+(n-2)^2+n^2)=n(n+1)(n+2)$$

Basis Step:

Assumption:

Induction Step:

Investigation 4 – The Sum of Odd numbers

Using a mathematical induction proof, show that for any positive integer n,

$$1 + 3 + 5 + . . . + (2n - 1) = n^2$$

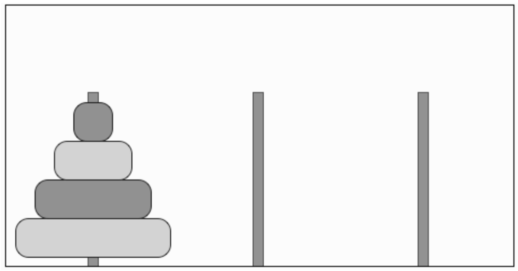
Basis Step:

Assumption:

Induction Step:

Towers of Hanoi – An Induction Fantasy

You find yourselves transported into another dimension. Located in front of you is a strange object that consists of round stones and pegs.



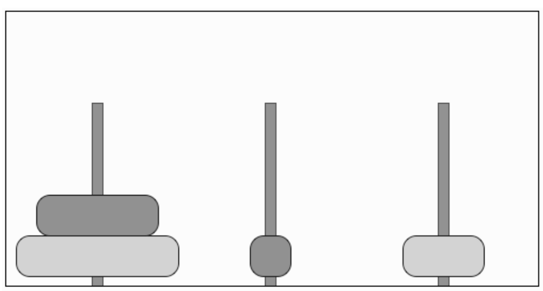
Suddenly a booming voice shouts,

"I am the God of Towers. You have been chosen as the mathematical savior of the world."

You immediately fall to the ground confused and beg for mercy.

The God of Towers continues:

"You are given a tower of four disks initially stacked in increasing size on one of three pegs. You must transfer the entire tower to one of the other pegs moving only one disk at a time and never a larger one onto a smaller."



You can view an interactive applet at:

http://www.cosc.canterbury.ac.nz/mukundan/dsal/ToHdb.html

In order to save yourself and the world you must complete the following four-part quest:

1 Find a relationship in the Towers of Hanoi puzzle that will predict the minimum number of moves for a set of rings, based on the minimum number of moves for one ring less (this is called a recursive relationship).

2 Find a relationship in the Towers of Hanoi puzzle that will predict the minimum number of moves for a set of rings, based solely upon the number of rings.

3 Sharpen your skills in mathematical induction.

4 Finally, save the world by using the recursive relationship in #1 to prove your conjecture in #2 by mathematical induction.

You start out by determining how many moves it will take to complete the task if the tower only consisted of two round stones. You can use the Tower of Hanoi applet to help you fill in the table below.

Number of Rings in the Tower	Minimum Number of Moves
2	3
3	7
4	15
5	
6	
7	
8	
. . . n	

1. Determine a relationship based on the number of moves